B.Sc. Part III (Hons.)

Lagrange's linear partial Diff. eqns

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Here, z is a function of x and y.

i.e. $z = z(x,y)$.

$\therefore\ p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} = z_x,\ \frac{\partial z}{\partial y} = z_y$

Form of Lagrange's equation

$$\boxed{Pp + Qq = R}$$

or $$\boxed{P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R}$$

Here, P, Q, R are functions of x, y, z.

AUXILIARY EQUATIONS

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{—— (A)}$$

We find two integrals from eq (A)

Let these be $u = a$ and $v = b$.

Then the general integral is

$f(u,v) = 0$ or $\phi(u) = v$

or $u = \phi(v)$.

Examples 1. $p+q=1$

Compare it with $Pp+Qq=R$

i.e. $P=1, Q=1, R=1$.

$\therefore$ Auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1}=\frac{dy}{1}=\frac{dz}{1}$$

$$\Rightarrow dx=dy=dz \quad \text{——} (A)$$

From (A), $dx=dy$. Integrating, we get

$$x=y+u \Rightarrow u=x-y \quad \text{——} (B)$$

From (B) $dy=dz$

Integrating, we get

$$y=z+v \Rightarrow v=y-z \quad \text{——} (C)$$

Note that eq. (B) and (C) are two integrals.

Hence, the general solution is

$$u=f(v)$$

$$\Rightarrow x-y=f(y-z)$$

$(y+z)p + (z+x)q = x+y.$

Soln

It is of the form

$Pp + Qq = R$

$\therefore P = y+z, \; Q = z+x, \; R = x+y.$

Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \longrightarrow (A)$$

Students $\longrightarrow$ find two integrals from (A).

from (A) $\dfrac{dx-dy}{(y+z)-(z+x)} = \dfrac{dy-dz}{(z+x)-(x+y)}$

$$\Rightarrow \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\Rightarrow \frac{dx-dy}{x-y} - \frac{dy-dz}{y-z} = 0$$

Integrating, we get

$\log(x-y) - \log(y-z) = \log u$

$$\Rightarrow u = \frac{x-y}{y-z}. \longrightarrow (B)$$

Again, from (A), $\dfrac{dx-dz}{(y+z)-(x+y)} = \dfrac{dx+dy+dz}{y+z+z+x+x+y}$

$$\Rightarrow \frac{dx+dy+dz}{2(x+y+z)} + \frac{dx-dz}{x-z} = 0$$

$$\Rightarrow \frac{dx+dy+dz}{x+y+z} + \frac{2(dx-dz)}{x-z} =$$

Integrating, we get

$$\log(x+y+z) + 2\log(x-z) = \log v$$

$$\Rightarrow \log(x+y+z) + \log\{(x-z)^2\} = \log v$$

$$\Rightarrow \log\left[(x+y+z)(x-z)^2\right] = \log v$$

$$\Rightarrow v = (x+y+z)(x-z)^2 \text{ ——— } (C)$$

Eq. (B) and (C) are two integral of the given eqn.

Hence the general soln of the given eqn is

$$u = \phi(v)$$

i.e. $\frac{x-y}{y-z} = \phi\left[(x+y+z)(x-z)^2\right]$

Solve $(a-x)p + (b-y)q = c-z$.

Soln

Here, $P = a-x$, $Q = b-y$, $R = c-z$

Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{a-x} = \frac{dy}{b-y} = \frac{dz}{c-z} \longrightarrow (A)$$

from (A), $\frac{dx}{a-x} = \frac{dy}{b-y}$. Integrating

$$-\log(a-x) = -\log(b-y) - \log u$$

$$\Rightarrow \log(a-x) = \log(b-y) + \log u$$

$$\Rightarrow u = \frac{a-x}{b-y}. \quad \longrightarrow (B)$$

from (B), $\frac{dx}{a-x} = \frac{dz}{c-z}$ integrating.

$$-\log(a-x) = -\log(c-z) + \log v$$

$$\Rightarrow v = \frac{a-x}{c-z} \longrightarrow (C)$$

from (B) and (C), $f\left(\frac{a-x}{b-y}, \frac{a-x}{c-z}\right) = 0$

$\boxed{Q}$ Solve $p\tan x + q\tan y = \tan z$

Soln Here $P = \tan x$, $Q = \tan y$, $R = \tan z$

$\therefore$ Auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$\Rightarrow \cot x\, dx = \cot y\, dy = \cot z\, dz$

$\therefore \cot x\, dx = \cot y\, dy$

$\Rightarrow \log \sin x = \log \sin y + u$

$\Rightarrow u = \frac{\sin x}{\sin y}$

Similarly $v = \frac{\sin y}{\sin z}$

So the soln is $f(U, V) = 0$

Q Solve

$zp - zq = z^2 + (x+y)^2.$

Soln Comparing it with $Pp + Qq = R$, we get

$P = z,\ Q = -z,\ R = z^2 + (x+y)^2$

$\therefore$ Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2} \quad \text{—— (A)}$$

From (A), $\frac{dx}{z} = \frac{dy}{-z}$

$\Rightarrow dx + dy = 0$. Integrating, we get

$\Rightarrow x + y = u$ —— (B)

From (A), $\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$

$$\Rightarrow dx = \frac{z\,dz}{z^2+(x+y)^2} = \frac{z\,dz}{z^2+u^2} \quad [\text{using (B)}]$$

Integrating, we get

$\Rightarrow 2x = \log(z^2+u^2) - v$

$\Rightarrow v = \log\{z^2+(x+y)^2\} - 2x$ —— (C)

Hence, the general integral is

$v = \phi(u) \Rightarrow \log\{z^2+(x+y)^2\} - 2x = \phi(x+y$